

Integral relations and new solutions to the double-confluent Heun equation

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Abstract

Integral relations and transformation rules are used to obtain, out of an asymptotic solution, a new group of four pairs of solutions to the double-confluent Heun equation. Each pair presents the same series coefficients but has solutions convergent in different regions of the complex plane. Integral relations are also established between solutions given by series of Coulomb wave functions. The Whittaker-Hill equation and another equation are studied as particular cases of both the double-confluent and the single-confluent Heun equations. Finally, applications for the Schrödinger equation with certain potentials are discussed, mainly for quasi-exactly solvable potentials which lead to the above special equations.

1. Preliminary remarks

Here we deal with two groups of series solutions to the double-confluent Heun equation (DCHE). The first group, constituted by four pairs of solutions, is generated from an asymptotic expansion by means of integral relations and transformation rules, and the second group is given by pairs of solutions in series of Coulomb wave functions, already derived in [1]. For the latter, we show that an integral relation also exists between the members of each pair, and we provide additional properties for the solutions. After this, we analyse two differential equations which are special cases of both the DCHE and the single-confluent Heun equation and, finally, we use some results to solve the Schrödinger equation for certain potentials. Before proceeding, we set down some conventions, present the procedures used to obtain the solutions and outline the structure of the paper.

For the DCHE we adopt the form

$$z^2 \frac{d^2 U}{dz^2} + (B_1 + B_2 z) \frac{dU}{dz} + (B_3 - 2\eta\omega z + \omega^2 z^2) U = 0, \quad (B_1 \neq 0, \quad \omega \neq 0) \quad (1)$$

where $z = 0$ and $z = \infty$ are irregular singularities. $B_1 = 0$ and/or $\omega = 0$ are excluded because, in these cases, the equation reduces to a confluent hypergeometric equation or to an equation with constant coefficients. Thus, if $B_1 = 0$ and $\omega \neq 0$, the substitutions

$$y = -2i\omega z, \quad U(z) = e^{-y/2} y^\alpha f(y), \quad \alpha^2 - (1 - B_2)\alpha + B_3 = 0 \quad (2a)$$

give the confluent hypergeometric equation

$$y \frac{d^2 f}{dy^2} + [(2\alpha + B_2) - y] \frac{df}{dy} - \left(i\eta + \alpha + \frac{B_2}{2} \right) f = 0 \quad (2b)$$

and, if $B_1 \neq 0$ and $\omega = 0$,

$$y = B_1/z, \quad U(z) = y^\beta g(y), \quad \beta^2 - (B_2 - 1)\beta + B_3 = 0 \Rightarrow \quad (3a)$$

$$y \frac{d^2 g}{dy^2} + [(2\beta + 2 - B_2) - y] \frac{dg}{dy} - \beta g = 0. \quad (3b)$$

If $B_1 = \omega = 0$, we find an equation with constant coefficients by taking $z = \exp y$.

Equation (1) can be obtained from the generalized spheroidal wave equation (GSWE) [2], which is also known as single-confluent Heun equation or, simply, confluent Heun equation [3]. Actually the GSWE in Leaver's form reads [4]

$$z(z - z_0) \frac{d^2 U}{dz^2} + (B_1 + B_2 z) \frac{dU}{dz} + [B_3 - 2\eta\omega(z - z_0) + \omega^2 z(z - z_0)] U = 0, \quad (\omega \neq 0) \quad (4)$$

where z_0 , B_i , η and ω are constants, $z = 0$ and $z = z_0$ are regular singular points while $z = \infty$ is an irregular singularity. For $z_0 = 0$ the preceding equation gives the DCHE and, when a limiting solution also exists, this one can be taken as a starting-point to generate a group of solutions to the DCHE. This procedure was applied by Leaver to find expansions in series of Coulomb wave functions for the DCHE, although these can as well be found otherwise [5]. Note that, although the DCHE was called 'confluent GSWE' in [1, 4], it is not derived from the GSWE by a process of confluence (see [6], chapter 4).

The transformation rules aforementioned result from variable substitutions that convert the DCHE into another version of itself. Thus, for a given solution $S(z)$,

$$S(z) := U(B_1, B_2, B_3; \omega, \eta; z), \quad (5)$$

(':= ' means 'equal by definition') we shall generate new solutions by using the following rules [1,3,6-8] — r_1 , r_2 , r_3 —

$$r_1 S(z) = e^{i\omega z + B_1/(2z)} z^{-i\eta - B_2/2} U(B'_1, B'_2, B'_3; \omega', \eta'; \vartheta), \quad (6)$$

$$r_2 S(z) = e^{B_1/z} z^{2-B_2} U(-B_1, 4 - B_2, B_3 + 2 - B_2; \omega, \eta; z), \quad (7)$$

$$r_3 S(z) = U(B_1, B_2, B_3; -\omega, -\eta; z), \quad (8)$$

where, on the right-hand side of the first relation, we have

$$\begin{aligned} B'_1 &= \omega B_1, \quad B'_2 = 2 + 2i\eta, \quad B'_3 = B_3 - \left(\frac{B_2}{2} + i\eta \right) \left(\frac{B_2}{2} - i\eta - 1 \right), \\ \omega' &= 1, \quad i\eta' = \frac{B_2}{2} - 1, \quad \vartheta = \frac{iB_1}{2z}. \end{aligned} \quad (9)$$

In the third rule it is assumed that we must change the sign of (η, ω) only where these quantities appear explicitly in $S(z)$, keeping the expressions for the other parameters unchanged, even if they depend on η and ω . Moreover, note that the transformation r_1 generally gives a solution having a region of convergence different from that of the initial solution, since it involves an inversion of the independent variable. For brevity, we use only r_1 and r_2 , although r_3 must be employed to furnish the full group of solutions.

An integral relation can also be used to generate a new solution from a previously known one. It provides a pair of solutions which has *essentially* the same series coefficients (that is, coefficients which differ at most by a constant not depending on n) but put some restrictions on the parameters and arguments of the solutions. The transformation rules applied to that pair afford new ones, having again the same coefficients and solutions connected to each other by integral relations.

We denote by U_i^∞ the series solutions which converge for $|z| > 0$, and by U_i^0 those which converge for $|z| < \infty$, the subscript i indicating the pair to which the solution belongs. If there is a phase parameter ν , it appears as a subscript as well, let us say, $U_{i\nu}^\infty$ and $U_{i\nu}^0$. For a specified pair, U_i^0 will result from U_i^∞ via an integral transformation. Solutions with a phase parameter are given by two-sided series in which the summation index n runs from $-\infty$ to ∞ , whereas solutions without that parameter are given by one-sided series ($n \geq 0$). Under certain conditions, the latter become finite-series solutions which are called quasi-polynomial solutions, Heun polynomials or quasi-algebraic solutions.

In the next section we find the kernels to the integral relations used in sections 3 and 4. In section 3, we use integral relations and transformations rules to derive four pairs of solutions from an asymptotic expansion in the vicinity of $z = \infty$. In each pair, one solution is given as an ascending or descending power series of z and the other as a series of irregular confluent hypergeometric functions which, for terminating series, may be written in terms of generalized Laguerre polynomials.

In section 4 we determine the integral relations between expansions in Coulomb wave functions. We use only expansions in series of irregular hypergeometric functions, and find that, in each pair without a phase parameter, one solution may again be expressed as a generalized Laguerre polynomial. In addition, we verify that finite-series solutions occur under the same conditions valid for the corresponding solutions in section 3.

In section 5 we obtain normal forms for the DCHE — in which there is no first derivative term — and examine the two differential equations which are particular cases of both the DCHE and the GSWE, namely,

$$\frac{d^2 W_1}{du^2} + [\theta_0 + \theta_1 \cosh(\kappa u) + \theta_2 \cosh(2\kappa u)] W_1 = 0, \quad (10a)$$

$$\frac{d^2 W_2}{du^2} + [\bar{\theta}_0 + \bar{\theta}_1 \sinh(\kappa u) + \bar{\theta}_2 \cosh(2\kappa u)] W_2 = 0, \quad (10b)$$

where the first equation represents the Whittaker-Hill equation (WHE) or the modified WHE depending on whether κu is pure imaginary or real, respectively. We also discuss the solutions to the Schrödinger equation for some potentials which give rise to DCHEs, and specially to these particular cases. Section 6 is devoted to final comments while, in the Appendix, integrals used to establish integral relations are written in terms of irregular confluent hypergeometric functions rather than in terms of Whittaker functions.

2. Kernels for integral relations

Several kernels are possible for integral transformations of the DCHE [5, 6], but we only regard those that will be useful in the subsequent sections. We follow a procedure similar to the one employed by Schmidt and Wolf [5], adapted to the form we have chosen for the DCHE. Thus, if $U(z)$ is a known solution of equation (1), we seek a new solution $\hat{U}(z)$ given by the integral relation

$$\hat{U}(z) = \int_{t_1}^{t_2} K(z, t) U(t) dt, \quad (11)$$

where the kernel $K(z, t)$ is determined from [9]

$$L_z\{K(z, t)\} = \overline{L}_t\{K(z, t)\}, \quad (12a)$$

being the operator L_z and its adjoint \overline{L}_z given by

$$\begin{aligned} L_z &:= z^2 \frac{\partial^2}{\partial z^2} + [B_1 + B_2 z] \frac{\partial}{\partial z} + (\omega^2 z^2 - 2\omega\eta z), \\ \overline{L}_t &:= t^2 \frac{\partial^2}{\partial t^2} + [-B_1 + (4 - B_2)t] \frac{\partial}{\partial t} + (\omega^2 t^2 - 2\omega\eta t + 2 - B_2). \end{aligned} \quad (12b)$$

In terms of L_z , equation (1) reads

$$[L_z + B_3]U(z) = 0, \quad (13)$$

where L_z is now understood as an ordinary differential operator. In equation (11) we have chosen the contour of integration as the line joining t_1 and t_2 , but we assume that these endpoints depend on z and, consequently, we have to use the formula

$$\frac{d}{dz} \int_{t_1}^{t_2} F(z, t) dt = \int_{t_1}^{t_2} \frac{\partial F(z, t)}{\partial z} dt + F(z, t_2) \frac{dt_2}{dz} - F(z, t_1) \frac{dt_1}{dz} \quad (14)$$

in order to derive the conditions under which $\hat{U}(z)$ is solution of the DCHE. Hence, applying L_z to integral (11) and using equations (12a) and (14), we find

$$L_z\{\hat{U}(z)\} = \int_{t_1}^{t_2} \overline{L}_t\{K(z, t)\}U(t)dt + Q(z, t_2) - Q(z, t_1), \quad (15a)$$

where $(i = 1, 2)$

$$\begin{aligned} Q(z, t_i) &:= \left[z^2 \frac{d^2 t_i}{dz^2} + (B_1 + B_2 z) \frac{dt_i}{dz} \right] U(t_i) K(z, t_i) + \\ & z^2 U(t_i) \left[\frac{\partial K(z, t_i)}{\partial t_i} \left(\frac{dt_i}{dz} \right)^2 + 2 \frac{\partial K(z, t_i)}{\partial z} \frac{dt_i}{dz} \right] + z^2 \left(\frac{dt_i}{dz} \right)^2 \frac{dU(t_i)}{dt_i} K(z, t_i). \end{aligned} \quad (15b)$$

The notation $\overline{L}_t\{K(z, t)\}U(t)$ in equation (15a) means that the operator \overline{L}_t acts uniquely on the object inside braces. Next, we integrate equation (15a) by parts or, equivalently, by using the identity

$$\overline{L}_t\{K(z, t)\}U(t) - K(z, t)L_t\{U(t)\} = \frac{\partial}{\partial t}P(z, t),$$

where $P(z, t)$ is given by

$$P(z, t) := t^2 \left[U(t) \frac{\partial K(z, t)}{\partial t} - K(z, t) \frac{\partial U(t)}{\partial t} \right] + [(2 - B_2)t - B_1] U(t) K(z, t). \quad (16)$$

Then we find

$$L_z\{\hat{U}(z)\} = \int_{t_1}^{t_2} \left[K(z, t)L_t\{U(t)\} + \frac{\partial P(z, t)}{\partial t} \right] dt + Q(z, t_2) - Q(z, t_1),$$

or, using equations (13) and (11),

$$(L_z + B_3)\hat{U}(z) = P(z, t_2) + Q(z, t_2) - P(z, t_1) - Q(z, t_1). \quad (17)$$

Therefore, when $U(z)$ is a solution of the DCHE, $\hat{U}(z)$ will also be a solution if the integral (11) exist and the right hand side of equation (17) vanishes. In fact, we require that the ‘integrated terms’ vanish when $t \rightarrow t_i$, that is,

$$Q(z, t_i) + P(z, t_i) = 0, \quad (18a)$$

where, due to equations (15b) and (16),

$$\begin{aligned} P(z, t_i) + Q(z, t_i) = & \left[z^2 \frac{d^2 t_i}{dz^2} + (B_1 + B_2 z) \frac{dt_i}{dz} + (2 - B_2) t_i - B_1 \right] K(z, t_i) U(t_i) + \left[z^2 \left(\frac{dt_i}{dz} \right)^2 - (t_i)^2 \right] \times \\ & K(z, t_i) \frac{dU(t_i)}{dt_i} + \left\{ 2z^2 \frac{\partial K(z, t_i)}{\partial z} \frac{dt_i}{dz} + \left[z^2 \left(\frac{dt_i}{dz} \right)^2 + (t_i)^2 \right] \frac{\partial K(z, t_i)}{\partial t_i} \right\} U(t_i). \end{aligned} \quad (18b)$$

Below, we find two kernels for integral relations and show that, for these, the right-hand side of (18b) assumes a very simple form. From these kernels, two others — and the corresponding transformed solutions $\hat{U}(z)$ — may be obtained by the change $(\eta, \omega) \rightarrow (-\eta, -\omega)$. These kernels are necessary to set up integral relations between the solutions derived from the ones considered in this paper by means of rule r_3 .

First kernel: Performing the substitution

$$K(z, t) = e^{i\omega(z+t)+(B_1/z)} z^{2-B_2} H_1(z, t) \quad (19)$$

in equation (12a) and supposing that H_1 depends on z and t by the product zt , we find

$$\begin{aligned} (\xi - 1) \frac{dH_1}{d\xi} - \left(\frac{B_2}{2} - i\eta - 2 \right) H_1 &= 0, \quad \xi := -2i\omega zt/B_1 \Rightarrow \\ H_1(z, t) &= (\xi - 1)^{(B_2/2)-i\eta-2} = \left(-\frac{2i\omega}{B_1} zt - 1 \right)^{\frac{B_2}{2}-i\eta-2}. \end{aligned} \quad (20)$$

Thus, we have a first kernel, denoted by $K_1(z, t)$, namely,

$$K_1(z, t) = e^{i\omega(z+t)+(B_1/z)} z^{2-B_2} \left(-\frac{2i\omega}{B_1} zt - 1 \right)^{\frac{B_2}{2}-i\eta-2} \quad (21)$$

which yields

$$\hat{U}_1(z) = e^{i\omega z+(B_1/z)} z^{2-B_2} \int_{t_1}^{t_2} dt \left[e^{i\omega t} \left(-\frac{2i\omega}{B_1} zt - 1 \right)^{\frac{B_2}{2}-i\eta-2} U(t) \right].$$

We rewrite this integral in terms of ξ and integrate from ξ_1 to ξ_2 , assuming that these new endpoints are constants to be specified later. We get

$$\hat{U}_1(z) = e^{i\omega z+(B_1/z)} z^{1-B_2} \int_{\xi_1}^{\xi_2} d\xi \left[e^{-B_1\xi/(2z)} (\xi - 1)^{(B_2/2)-i\eta-2} U \left(-\frac{B_1\xi}{2i\omega z} \right) \right]. \quad (22)$$

On the other hand, by inserting

$$t_i(z) = -B_1\xi_i/(2i\omega z) \quad (23a)$$

and $K(z, t) = K_1(z, t)$ into the right-hand side of the expression (18b), we find that the first term becomes

$$\left[\frac{(B_1)^2 \xi_i}{2i\omega z^2} + \frac{B_1 \xi_i (B_2 - 2)}{i\omega z} - B_1 \right] K_1(z, t_i) U(t_i),$$

the second term vanishes because $z^2 (dt_i/dz)^2 - (t_i)^2 = 0$, and the last term reduces to

$$\left[\frac{(B_1)^2 \xi_i (\xi_i - 2)}{2i\omega z^2} - \frac{B_1 \xi_i (B_2 - 2)}{i\omega z} + B_1 \xi_i \right] K_1(z, t_i) U(t_i).$$

Therefore we can write

$$\begin{aligned} P_1(z, t_i) + Q_1(z, t_i) &= \left[\frac{(B_1)^2 \xi_i}{2i\omega z^2} + B_1 \right] (\xi_i - 1) K_1(z, t_i) U(t_i) \\ &= \left[\frac{(B_1)^2 \xi_i}{2i\omega z^2} + B_1 \right] z^{2-B_2} (\xi_i - 1)^{(B_2/2)-i\eta-1} U(t_i) \exp \left[i\omega z + \frac{B_1}{z} - \frac{B_1 \xi_i}{2z} \right]. \end{aligned} \quad (23b)$$

Now we choose $\xi_1 = 1$ and $\xi_2 = \infty$ and then we can use one of the integrals given in the Appendix to integrate equation (22) for the solutions given in sections 3 and 4.

Second kernel: Accomplishing the substitution

$$K(z, t) = e^{i\omega(z+t)-(B_1/t)} t^{B_2-2} H_2(z, t) \quad (24)$$

in equation (12a) and supposing that $H_2(z, t)$ depends on z and t by the product zt , we find

$$\begin{aligned} (\zeta - 1) \frac{dH_2}{d\zeta} + \left(i\eta + \frac{B_2}{2} \right) H_2 &= 0, \quad \zeta := 2i\omega zt/B_1 \Rightarrow \\ H_2(z, t) &= (\zeta - 1)^{-(B_2/2)-i\eta} = \left(\frac{2i\omega}{B_1} zt - 1 \right)^{-\frac{B_2}{2}-i\eta}. \end{aligned} \quad (25)$$

Hence, the second kernel, $K_2(z, t)$, is

$$K_2(z, t) = e^{i\omega(z+t)-(B_1/t)} t^{B_2-2} \left(\frac{2i\omega}{B_1} zt - 1 \right)^{-\frac{B_2}{2}-i\eta} \quad (26)$$

which implies

$$\hat{U}_2(z) = e^{i\omega z} \int_{t_1}^{t_2} dt \left[e^{i\omega t - (B_1/t)} t^{B_2-2} \left(\frac{2i\omega}{B_1} zt - 1 \right)^{-\frac{B_2}{2}-i\eta} U(t) \right],$$

or, in terms of ζ ,

$$\hat{U}_2(z) = e^{i\omega z} z^{1-B_2} \int_{\zeta_1}^{\zeta_2} d\zeta \left[e^{B_1 \zeta / (2z) - 2i\omega z / \zeta} \zeta^{B_2-2} (\zeta - 1)^{-(B_2/2)-i\eta} U \left(\frac{B_1 \zeta}{2i\omega z} \right) \right]. \quad (27)$$

This time we have

$$t_i = B_1 \zeta_i / (2i\omega z) \quad (28a)$$

in the condition (18a), and equation (18b) can be rewritten as

$$P_2(z, t_i) + Q_2(z, t_i) = \left[\frac{(B_1)^2 \zeta_i}{2i\omega z^2} - B_1 \right] (\zeta_i - 1) K_2(z, t_i) U(t_i) = \left[\frac{(B_1)^2 \zeta_i}{2i\omega z^2} - B_1 \right] \times \left(\frac{B_1 \zeta_i}{2i\omega z} \right)^{B_2-2} (\zeta_i - 1)^{1-(B_2/2)-i\eta} U(t_i) \exp \left[i\omega z + \frac{B_1 \zeta_i}{2z} - \frac{2i\omega z}{\zeta_i} \right]. \quad (28b)$$

We choose $\zeta_1 = 1$ and $\zeta_2 = \infty$ and, then, we can once more use one of the integrals of the Appendix to integrate equation (27).

The approach we have used in this section is similar to the one employed by Schmidt and Wolf [5], in which we have regarded ξ or ζ as integration variables instead of t . This, in turn, implies that t_1 and t_2 in the the integral (11) are functions of z , as far as the integration endpoints in the variables ξ or ζ are taken as constants throughout sections 3-4. As said before, we shall find $U(t) = U^\infty(t)$ and $\hat{U}(z) = U^0(z)$ in the integral relations. Then, using the first identification together with the validity conditions for the integrals, it will be easy to check that the right-hand sides of equations (23b) and (28b) vanish for the solutions given in sections 3 and 4.

3. Solutions derived from an asymptotic expansion

The simpler quasi-polynomial solutions to the DCHE are obtained from the asymptotic expansions in the vicinity of the singular points 0 or ∞ [5]. In this section, we begin with an asymptotic representation for $z \rightarrow \infty$ and use the integral relations and transformation formulae to form four pairs of solutions. In each pair, one solution is given in terms of a series of ascending or descending powers of z , and the other in terms of a series of irregular confluent hypergeometric functions. We find the conditions for obtaining Heun polynomials and verify that, in this case, the hypergeometric functions degenerate to generalized Laguerre polynomials.

The starting-point solution is given by

$$U_1^\infty(z) = e^{i\omega z} z^{-i\eta-(B_2/2)} \sum_{n=0}^{\infty} b_n^{(1)} (-2i\omega z)^{-n}, \quad (29a)$$

where the recurrence relations for the coefficients $b_n^{(1)}$ are, in abbreviated notation,

$$\alpha_0 b_1 + \beta_0 b_0 = 0, \quad \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \quad (n \geq 1), \quad (29b)$$

in which

$$\begin{aligned} \alpha_n &= n + 1, \\ \beta_n &= n(n + 1 + 2i\eta) + i\omega B_1 + B_3 + \left(\frac{B_2}{2} + i\eta \right) \left(1 + i\eta - \frac{B_2}{2} \right), \\ \gamma_n &= 2i\omega B_1 [n + i\eta + (B_2/2) - 1]. \end{aligned} \quad (29c)$$

These relations yield a characteristic equation in terms of the infinite continued fraction

$$\beta_0 = \frac{\alpha_0 \gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \frac{\alpha_2 \gamma_3}{\beta_3 -} \dots. \quad (30)$$

To obtain the foregoing recurrence relations, we perform the substitutions

$$U_1^\infty(z) = e^{i\omega z} z^{-i\eta-(B_2/2)} Y(y), \quad y = (-2i\omega z)^{-1},$$

and find

$$y^2 \frac{d^2 Y}{dy^2} + \left[1 + (2 + 2i\eta)y + 2i\omega B_1 y^2 \right] \frac{dY}{dy} + \left[C + 2i\omega B_1 \left(i\eta + \frac{B_2}{2} \right) y \right] Y = 0,$$

$$C := i\omega B_1 + B_3 + \left(\frac{B_2}{2} + i\eta \right) \left(1 + i\eta - \frac{B_2}{2} \right).$$

Then, inserting $Y(y) = \sum_{n=0}^{\infty} b_n^{(1)} y^n$ into this equation and proceeding in the usual form, we obtain the previous relations.

According to the theory of ordinary differential equations in the complex domain (see [10], chapter 7), the solution $U_1^\infty(z)$ is unique (one-valued) within the sector

$$-\frac{3\pi}{2} < \arg(-2i\omega z) < \frac{3\pi}{2}. \quad (31)$$

However we still have to show that it converges for $|z| > 0$. To accomplish this, we divide the recurrence relations (29b) by $b_n^{(1)}$ and take the limit when $n \rightarrow \infty$. This gives

$$\lim_{n \rightarrow \infty} \frac{b_{n+1}^{(1)}}{b_n^{(1)}} = -\frac{2i\omega B_1}{n}, \text{ or } \lim_{n \rightarrow \infty} \frac{b_{n+1}^{(1)}}{b_n^{(1)}} = -n. \quad (32)$$

These limits may as well be derived by using a Perron-Kreuser theorem for difference equations [11]. In order to satisfy characteristic equation (30), we have to choose the first limit (minimal solution). Thence,

$$\lim_{n \rightarrow \infty} \frac{b_{n+1}^{(1)} (-2i\omega z)^{-n-1}}{b_n^{(1)} (-2i\omega z)^{-n}} = \frac{B_1}{nz}$$

and, therefore, $U_1^\infty(z)$ converges for any $|z| > 0$.

On the other hand, to get a solution $U_1^0(z)$ convergent in the neighborhood of $z = 0$ we insert $U_1^\infty(t)$ into the integral relation (22). This gives

$$U_1^0(z) := \hat{U}_1(z) \propto e^{i\omega z + (B_1/z)} z^{1+i\eta - (B_2/2)} \sum_{n=0}^{\infty} b_n^{(1)} \left(\frac{z}{B_1} \right)^n I_n^{(1)}(z),$$

being

$$I_n^{(1)}(z) := \int_1^\infty d\xi \left[e^{-B_1 \xi / z} (\xi - 1)^{(B_2/2) - i\eta - 2} \xi^{-n - i\eta - (B_2/2)} \right] \stackrel{A1}{=} \\ \Gamma \left(\frac{B_2}{2} - i\eta - 1 \right) e^{-B_1/z} \left(\frac{B_1}{z} \right)^{1+n+2i\eta} U \left(n + i\eta + \frac{B_2}{2}, n + 2 + 2i\eta, \frac{B_1}{z} \right)$$

where we have used integral (A1) which is valid if

$$\Re[(B_2/2) - i\eta - 1] > 0, \quad \Re(B_1/z) > 0. \quad (33)$$

Thus, the sought solution is given, apart from a multiplicative factor, by

$$U_1^0(z) = e^{i\omega z} z^{-i\eta - (B_2/2)} \sum_{n=0}^{\infty} b_n^{(1)} U \left(n + i\eta + \frac{B_2}{2}, n + 2 + 2i\eta, \frac{B_1}{z} \right), \quad (34)$$

where $U(a, b, y)$ denotes the irregular confluent hypergeometric function [12]. Note moreover that, by inserting U_1^∞ into equation (23b), we have

$$P_1(z, t_i) + Q_1(z, t_i) \propto \left[\frac{(B_1)^2 \xi_i}{2i\omega z^2} + B_1 \right] z^{2+i\eta-(B_2/2)} \xi_i^{-i\eta-(B_2/2)} (\xi_i - 1)^{(B_2/2)-i\eta-1} \\ \times \exp \left[i\omega z - \frac{B_1}{z} (\xi_i - 1) \right] \sum_{n=0}^{\infty} b_n^{(1)} \left(\frac{B_1 \xi_i}{z} \right)^{-n}.$$

The series on the right-hand side converges at $\xi_1 = 1$ and at $\xi_2 = \infty$. Then, the condition (33) assures that, for $\xi_1 = 1$, the second member goes to zero since $(\xi_1 - 1)^{(B_2/2)-i\eta-1} \rightarrow 0$; for $\xi_2 = \infty$, the second member also vanishes because $\exp[-B_1(\xi_2 - 1)/z] \rightarrow 0$. Arguments similar to these may be repeated for the other pairs of solutions written below.

To obtain the behaviour of $U_1^0(z)$ when $z \rightarrow 0$, we use the relation [13]

$$U(a, b, y) \sim y^{-a} [1 + O(|y|^{-1})], \quad -3\pi/2 < \arg y < 3\pi/2, \quad (|y| \rightarrow \infty), \quad (35)$$

thereof we find

$$\lim_{z \rightarrow 0} U_1^0(z) \sim 1 + O\left(\frac{z}{B_1}\right), \quad \text{within the sector } -\frac{3\pi}{2} < \arg\left(\frac{B_1}{z}\right) < \frac{3\pi}{2}. \quad (36)$$

However, to show that the series in $U_1^0(z)$ converges for $|z| < \infty$, we must consider the behaviour of $U(a, b, z)$ when $b \rightarrow \infty$, while $b - a$ and z remain bounded [13]. In this manner we get

$$\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n} = \frac{z}{B_1}, \quad f_n := U\left(n + i\eta + \frac{B_2}{2}, n + 2 + 2i\eta, \frac{B_1}{z}\right).$$

Then, combining this expression with the first limit given in (32), we have

$$\lim_{n \rightarrow \infty} \frac{b_{n+1}^{(1)} f_{n+1}}{b_n^{(1)} f_n} = -\frac{2i\omega z}{n}. \quad (37)$$

Therefore, $U_1^0(z)$ converges in any finite region of the complex plane. In case of finite series, the ratio test becomes meaningless and the convergence must be decided by inspection.

Starting from the first pair of solutions, we generate three others by the transformation rules r_1 and r_2 and, in each pair, the solutions are also connected by an integral transformation. Below, we collect up the four pairs of solutions (U_i^∞, U_i^0) ($i = 1, 2, 3, 4$), the validity conditions for the integral transformations, the asymptotic behaviour of each solution, and the sufficient condition to obtain quasi-polynomial solutions. For the latter solutions, the functions $U(a, b, z)$ degenerate to an generalized Laguerre polynomials because the parameter a becomes a negative integer $-l$, and thus we have [12]

$$U(-l, 1 + \alpha, y) = (-1)^l l! L_l^\alpha(y). \quad (38)$$

The condition for quasi-polynomial solutions results from the fact that a series with three-term recurrence relations such as (29b) becomes a finite series with $0 \leq n \leq N - 1$ if $\gamma_n = 0$ for some $n = N = \text{positive integer}$ [14].

First pair : (U_1^∞, U_1^0) .

$$U_1^\infty(z) = e^{i\omega z} z^{-i\eta-(B_2/2)} \sum_{n=0}^{\infty} b_n^{(1)} (-2i\omega z)^{-n}, \quad (39a) \\ U_1^0(z) = e^{i\omega z} z^{-i\eta-(B_2/2)} \sum_{n=0}^{\infty} b_n^{(1)} U\left(n + i\eta + \frac{B_2}{2}, n + 2 + 2i\eta, \frac{B_1}{z}\right).$$

$$\alpha_n^{(1)} = n + 1,$$

$$\beta_n^{(1)} = n(n + 1 + 2i\eta) + i\omega B_1 + B_3 + \left(\frac{B_2}{2} + i\eta\right) \left(1 + i\eta - \frac{B_2}{2}\right), \quad (39b)$$

$$\gamma_n^{(1)} = 2i\omega B_1 \left(n + i\eta + \frac{B_2}{2} - 1\right).$$

$$\text{Integral relation (22): } \Re\left(\frac{B_2}{2} - i\eta - 1\right) > 0 \text{ and } \Re\left(\frac{B_1}{z}\right) > 0. \quad (39c)$$

$$\text{Finite series: } (B_2/2) + i\eta = 1 - N \Rightarrow 0 \leq n \leq N - 1. \quad (39d)$$

To find the integral relation we have used equation (A1). The asymptotic behaviours are

$$\lim_{z \rightarrow \infty} U_1^\infty(z) \sim e^{i\omega z} z^{-i\eta - (B_2/2)}, \quad -\frac{3\pi}{2} < \arg(-2i\omega z) < \frac{3\pi}{2}; \quad (40a)$$

$$\lim_{z \rightarrow 0} U_1^0(z) \sim 1, \quad -\frac{3\pi}{2} < \arg\left(\frac{B_1}{z}\right) < \frac{3\pi}{2}. \quad (40b)$$

Second pair : $U_2^\infty \propto r_2 U_1^\infty$ and $U_2^0 \propto r_2 U_1^0$.

$$U_2^\infty(z) = e^{i\omega z + (B_1/z)} z^{-i\eta - (B_2/2)} \sum_{n=0}^\infty b_n^{(2)} (-2i\omega z)^{-n}, \quad (41a)$$

$$U_2^0(z) = e^{i\omega z + (B_1/z)} z^{-i\eta - (B_2/2)} \sum_{n=0}^\infty b_n^{(2)} U\left(n + 2 + i\eta - \frac{B_2}{2}, n + 2 + 2i\eta, -\frac{B_1}{z}\right).$$

$$\alpha_n^{(2)} = n + 1,$$

$$\beta_n^{(2)} = n(n + 1 + 2i\eta) - i\omega B_1 + B_3 + \left(\frac{B_2}{2} + i\eta\right) \left(1 + i\eta - \frac{B_2}{2}\right), \quad (41b)$$

$$\gamma_n^{(2)} = -2i\omega B_1 \left(n + 1 + i\eta - \frac{B_2}{2}\right).$$

$$\text{Integral relation (27): } \Re\left(\frac{B_2}{2} + i\eta - 1\right) < 0 \text{ and } \Re\left(\frac{B_1}{z}\right) < 0. \quad (41c)$$

$$\text{Finite series, if } (B_2/2) - i\eta = 1 + N \Rightarrow 0 \leq n \leq N - 1. \quad (41d)$$

The integral relation is found by using equation (A1). The asymptotic expressions are

$$\lim_{z \rightarrow \infty} U_2^\infty(z) \sim e^{i\omega z} z^{-i\eta - (B_2/2)}, \quad -\frac{3\pi}{2} < \arg(-2i\omega z) < \frac{3\pi}{2}; \quad (42a)$$

$$\lim_{z \rightarrow 0} U_2^0(z) \sim e^{B_1/z} z^{2 - B_2}, \quad -\frac{3\pi}{2} < \arg\left(-\frac{B_1}{z}\right) < \frac{3\pi}{2}. \quad (42b)$$

Third pair : $U_3^\infty \propto r_1 U_1^0$ and $U_3^0 \propto r_1 U_1^\infty$.

$$U_3^\infty(z) = e^{i\omega z} \sum_{n=0}^\infty b_n^{(3)} U\left(n + i\eta + \frac{B_2}{2}, n + B_2, -2i\omega z\right), \quad (43a)$$

$$U_3^0(z) = e^{i\omega z} \sum_{n=0}^\infty b_n^{(3)} \left(\frac{z}{B_1}\right)^n.$$

$$\alpha_n^{(3)} = n + 1,$$

$$\beta_n^{(3)} = n(n + B_2 - 1) + i\omega B_1 + B_3, \quad (43b)$$

$$\gamma_n^{(3)} = 2i\omega B_1 \left(n + i\eta + \frac{B_2}{2} - 1\right).$$

$$\text{Integral relation (22): } \Re\left(\frac{B_2}{2} - i\eta - 1\right) > 0 \text{ and } \Re\left(\frac{B_1}{z}\right) > 0. \quad (43c)$$

$$\text{Finite series, if } (B_2/2) + i\eta = 1 - N \Rightarrow 0 \leq n \leq N - 1. \quad (43d)$$

To find the integral relation we use the equation (A2). The asymptotic behaviours are the same as in the first pair, that is,

$$\lim_{z \rightarrow \infty} U_3^\infty(z) \sim e^{i\omega z} z^{-i\eta - (B_2/2)}, \quad -\frac{3\pi}{2} < \arg(-2i\omega z) < \frac{3\pi}{2}; \quad (44a)$$

$$\lim_{z \rightarrow 0} U_3^0(z) \sim 1, \quad -\frac{3\pi}{2} < \arg\left(\frac{B_1}{z}\right) < \frac{3\pi}{2}. \quad (44b)$$

Fourth pair : $U_4^\infty \propto r_2 U_3^\infty$ and $U_4^0 \propto r_2 U_3^0$.

$$U_4^\infty(z) = e^{i\omega z + (B_1/z)} z^{2-B_2} \sum_{n=0}^\infty b_n^{(4)} U\left(n+2+i\eta-\frac{B_2}{2}, n+4-B_2, -2i\omega z\right), \quad (45a)$$

$$U_4^0(z) = e^{i\omega z + (B_1/z)} z^{2-B_2} \sum_{n=0}^\infty b_n^{(4)} \left(-\frac{z}{B_1}\right)^n.$$

$$\begin{aligned} \alpha_n^{(4)} &= n+1, \\ \beta_n^{(4)} &= n(n+3-B_2) + 2 - i\omega B_1 - B_2 + B_3, \end{aligned} \quad (45b)$$

$$\gamma_n^{(4)} = -2i\omega B_1 \left(n+1+i\eta-\frac{B_2}{2}\right). \quad (45c)$$

$$\text{Integral relation (27): } \Re\left(\frac{B_2}{2} + i\eta - 1\right) < 0 \text{ and } \Re\left(\frac{B_1}{z}\right) < 0. \quad (45c)$$

$$\text{Finite series, if } (B_2/2) - i\eta = 1 + N \Rightarrow 0 \leq n \leq N-1. \quad (45d)$$

To find the integral transformation, we must use relation (A2) again. The asymptotic behaviours are the same as in the second pair, namely,

$$\lim_{z \rightarrow \infty} U_4^\infty(z) \sim e^{i\omega z} z^{-i\eta - (B_2/2)}, \quad -\frac{3\pi}{2} < \arg(-2i\omega z) < \frac{3\pi}{2}, \quad (46a)$$

$$\lim_{z \rightarrow 0} U_4^0(z) \sim e^{B_1/z} z^{2-B_2}, \quad -\frac{3\pi}{2} < \arg\left(-\frac{B_1}{z}\right) < \frac{3\pi}{2}. \quad (46b)$$

For $z \rightarrow 0$, we have found the two asymptotic behaviours we could expect from the theory of differential equations. However, for $z \rightarrow \infty$, we have only one of the expected expressions. This occurs because we have regarded only one half of the solutions. In effect, if we apply rule r_3 to the preceding solutions, we get four new solutions $U_i^\infty(z)$ ($i = 5, \dots, 8$) for which

$$\lim_{z \rightarrow \infty} U_i^\infty(z) \sim e^{-i\omega z} z^{i\eta - (B_2/2)}, \quad -\frac{3\pi}{2} < \arg(2i\omega z) < \frac{3\pi}{2}.$$

Note as well that $U_3^0(z)$ and $U_4^0(z)$ can be reexpressed in terms of confluent hypergeometric functions. Indeed, if we use integral (A2) to derive these solutions from $U_3^\infty(z)$ and $U_4^\infty(z)$, we find

$$\begin{aligned} U_3^0(z) &\propto e^{i\omega z} z^{-i\eta - (B_2/2)} \sum_{n=0}^\infty b_n^{(3)} U\left(n+i\eta+\frac{B_2}{2}, n+1+i\eta+\frac{B_2}{2}, \frac{B_1}{z}\right), \\ U_4^0(z) &\propto e^{i\omega z + (B_1/z)} z^{-i\eta - (B_2/2)} \sum_{n=0}^\infty b_n^{(4)} U\left(n+2+i\eta-\frac{B_2}{2}, n+3+i\eta-\frac{B_2}{2}, -\frac{B_1}{z}\right), \end{aligned}$$

which are consistent with the previous expressions due to the transformation [12]

$$U(a, b, y) = z^{1-b} U(1+a-b, 2-b, y) \quad (47)$$

followed by (38) with $l = 0$. From the above expression for $U_3^0(z)$, we find

$$U_1^\infty(z) = r_1 U_3^0(z) \propto e^{i\omega z} \sum_{n=0}^{\infty} b_n^{(1)} U\left(n + i\eta + \frac{B_2}{2}, n + 1 + i\eta + \frac{B_2}{2}, -2i\omega z\right) \Rightarrow$$

$$U_2^\infty(z) = r_2 U_1^\infty(z) \propto e^{i\omega z + (B_1/z)} z^{2-B_2} \sum_{n=0}^{\infty} b_n^{(2)} \times$$

$$U\left(n + 2 + i\eta - \frac{B_2}{2}, n + 3 + i\eta - \frac{B_2}{2}, -2i\omega z\right),$$

which are alternative forms for $U_1^\infty(z)$ and $U_2^\infty(z)$. It is useful to write the two solutions in each pair as series of hypergeometric functions because we can use equation (35) to deduce the asymptotic behaviours of both solutions as well as the respective sectors inside which they are one-valued.

4. Solutions in series of Coulomb wave functions

In this section we establish integral relations for expansions in series of Coulomb wave functions (already found in section 4 of [1]) and provide some additional properties for these solutions. We consider only expansions in series of irregular confluent hypergeometric functions and, thus, we use the notation (U_i^∞, U_i^0) where we have used (\tilde{U}_i, U_i) in [1]. In effect, the solutions in terms of $U(a, b, z)$ afford the expected behaviour for the solutions when $z \rightarrow \infty$ and $1/z \rightarrow \infty$. Moreover, in section 4.2 we note that, in each pair of solutions without a phase parameter, one solution may be expressed in series of generalized Laguerre polynomials, and we also find that the conditions for quasi-polynomial solutions are the same as in the corresponding solutions of section 3. Therefore, to discard the expansions in regular confluent hypergeometric functions does not imply that we are setting aside finite-series solutions.

4.1. Solutions with a phase parameter

The first pair below is equivalent to the solutions found by Leaver [4]. $U_{1\nu}^0(z)$ can be derived from $U_{1\nu}^\infty(z)$ by the rule r_1 and also by an integral transformation. The second pair results from the first one by means of the rule r_2 ; its solutions are connected to one another by an integral relation but not by the rule r_1 .

First pair :

$$U_{1\nu}^\infty(z) = e^{i\omega z} z^{\nu+1-(B_2/2)} \sum_{n=-\infty}^{\infty} b_n (-2i\omega z)^n U(n + \nu + 1 + i\eta, 2n + 2\nu + 2, -2i\omega z),$$

$$U_{1\nu}^0(z) = e^{i\omega z} z^{-\nu-(B_2/2)} \sum_{n=-\infty}^{\infty} b_n \left(\frac{B_1}{z}\right)^n U\left(n + \nu + \frac{B_2}{2}, 2n + 2\nu + 2, \frac{B_1}{z}\right),$$
(48a)

with the following recurrence relations for the coefficients b_n

$$\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0, \quad (48b)$$

where

$$\alpha_n = \frac{i\omega B_1 [n + \nu + 2 - (B_2/2)] [n + \nu + 1 - i\eta]}{2[n + \nu + 1] [n + \nu + (3/2)]},$$

$$\beta_n = B_3 + \left(n + \nu + 1 - \frac{B_2}{2}\right) \left(n + \nu + \frac{B_2}{2}\right) + \frac{\eta\omega B_1 [(B_2/2) - 1]}{(n + \nu)(n + \nu + 1)},$$
(48c)

$$\gamma_n = \frac{i\omega B_1 [n + \nu + (B_2/2) - 1] [n + \nu + i\eta]}{2[n + \nu] [n + \nu - (1/2)]}.$$

$$\text{Integral relation (22): } \Re[(B_2/2) - i\eta - 1] > 0, \quad \Re(B_1/z) > 0. \quad (48d)$$

The phase parameter ν may be determined from a characteristic equation given as a sum of two infinite continued fractions, namely,

$$\beta_0 = \frac{\alpha_{-1}\gamma_0}{\beta_{-1}-} \frac{\alpha_{-2}\gamma_{-1}}{\beta_{-2}-} \frac{\alpha_{-3}\gamma_{-2}}{\beta_{-3}-} \dots + \frac{\alpha_0\gamma_1}{\beta_1-} \frac{\alpha_1\gamma_2}{\beta_2-} \frac{\alpha_2\gamma_3}{\beta_3-} \dots. \quad (49)$$

To show that these solutions are connected to each other by the integral relation mentioned above, we introduce $U(t) = U_{1\nu}^\infty(t)$ into the integral (22). Then we find

$$\hat{U}_1(z) \propto e^{i\omega z + (B_1/z)} z^{-\nu - (B_2/2)} \sum_{n=-\infty}^{\infty} b_n \left(\frac{B_1}{z} \right)^n I_n^{(1)}(z) \propto U_{1\nu}^0(z)$$

since $(\xi := -2i\omega z t / B_1)$

$$\begin{aligned} I_n^{(1)} &= \int_1^\infty e^{-B_1 \xi / z} (\xi - 1)^{(B_2/2) - i\eta - 2} \xi^{n+\nu+1-(B_2/2)} U \left(n + \nu + 1 + i\eta, 2n + 2\nu + 2, \frac{B_1}{z} \xi \right) d\xi \\ &\stackrel{A3}{=} \Gamma \left(\frac{B_2}{2} - i\eta - 1 \right) e^{-B_1/z} U \left(n + \nu + \frac{B_2}{2}, 2n + 2\nu + 2, \frac{B_1}{z} \right), \end{aligned}$$

under the conditions written in (48d). Furthermore, by inserting $U_{1\nu}^\infty(z)$ into condition (23b), we find $(t_i = -B_1 \xi_i / (2i\omega z))$

$$\begin{aligned} P_1(z, t_i) + Q_1(z, t_i) &\propto \left[\frac{(B_1)^2 \xi_i}{2i\omega z^2} + B_1 \right] z^{1-\nu-(B_2/2)} \xi_i^{1+\nu-(B_2/2)} (\xi_i - 1)^{(B_2/2) - i\eta - 1} \\ &\times \exp \left[i\omega z - \frac{B_1}{z} (\xi_i - 1) \right] \sum_{n=0}^{\infty} b_n^{(1)} \left(\frac{B_1 \xi_i}{z} \right)^n U \left(n + \nu + 1 + i\eta, 2n + 2\nu + 2, \frac{B_1}{z} \xi_i \right). \end{aligned}$$

As in section 3, the right-hand side of this expression vanishes for $\xi_1 = 1$ and $\xi_2 = \infty$ and, therefore, the condition (18a) is satisfied.

Second pair :

$$\begin{aligned} U_{2\nu}^\infty(z) &= f(z) z^{\nu+1-(B_2/2)} \sum_{n=-\infty}^{\infty} b'_n (-2i\omega z)^n U(n + \nu + 1 + i\eta, 2n + 2\nu + 2, -2i\omega z), \\ U_{2\nu}^0(z) &= f(z) z^{-\nu-(B_2/2)} \sum_{n=-\infty}^{\infty} b'_n \left(-\frac{B_1}{z} \right)^n U \left(n + \nu + 2 - \frac{B_2}{2}, 2n + 2\nu + 2, -\frac{B_1}{z} \right), \quad (50a) \\ f(z) &:= e^{i\omega z + (B_1/z)}, \end{aligned}$$

where the recurrence relations for b'_n are

$$\alpha'_n b'_{n+1} + \beta'_n b'_n + \gamma'_n b'_{n-1} = 0, \quad (50b)$$

with

$$\begin{aligned} \alpha'_n &= \frac{i\omega B_1 [n+\nu+(B_2/2)][n+\nu+1-i\eta]}{2[n+\nu+1][n+\nu+(3/2)]}, \\ \beta'_n &= -B_3 - \left(n + \nu + 1 - \frac{B_2}{2} \right) \left(n + \nu + \frac{B_2}{2} \right) - \frac{\eta\omega B_1 [(B_2/2)-1]}{(n+\nu)(n+\nu+1)}, \quad (50c) \\ \gamma'_n &= \frac{i\omega B_1 [n+\nu+1-(B_2/2)][n+\nu+i\eta]}{2[n+\nu][n+\nu-(1/2)]}. \end{aligned}$$

$$\text{Integral relation (27): } \Re[(B_2/2) + i\eta - 1] < 0, \quad \Re(B_1/z) < 0. \quad (50d)$$

The characteristic equation is analogous to (49). Again, it is simple to find the integral relation stated above. We insert $U_{2\nu}^\infty(t)$ into the right-hand side of relation (27) and get

$$\hat{U}_2(z) \propto e^{i\omega z} z^{-\nu-(B_2/2)} \sum_{n=-\infty}^{\infty} b'_n \left(-\frac{B_1}{z} \right)^n I_n^{(2)}(z) \propto U_{2\nu}^0(z),$$

since $(\zeta := 2i\omega zt/B_1)$

$$\begin{aligned} I_n^{(2)} &= \int_1^\infty e^{B_1\zeta/z} (\zeta - 1)^{-(B_2/2)-i\eta} \zeta^{n+\nu-1+(B_2/2)} U\left(n + \nu + 1 + i\eta, 2n + 2\nu + 2, -\frac{B_1}{z}\zeta\right) d\zeta \\ &\stackrel{A3}{=} \Gamma\left(1 - i\eta - \frac{B_2}{2}\right) e^{B_1/z} U\left(n + \nu + 2 - \frac{B_2}{2}, 2n + 2\nu + 2, -\frac{B_1}{z}\right). \end{aligned}$$

On the other hand, by inserting $U_{2\nu}^\infty$ into the equation (28b) we find that the condition (18a) is satisfied. Moreover, we note that the validity conditions for the integrals do not involve the phase parameter and, therefore, these integral relations are also valid for the truncated solutions.

4.2. Solutions without phase parameter

These come from the truncation of the solutions with phase parameter ($n \geq 0$) but, contrary to the solutions given in section 3, now there are three possible forms for the recurrence relations and for the corresponding characteristic equations (see appendix of [1]). For completeness, we write out these relations. In the first one we have $\alpha_{-1} = 0$ and, in the other cases, α_{-1} may be different from zero.

$$\left. \begin{aligned} \alpha_0 b_1 + \beta_0 b_0 &= 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} &= 0 \quad (n \geq 1), \end{aligned} \right\} \Rightarrow \beta_0 = \frac{\alpha_0 \gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \frac{\alpha_2 \gamma_3}{\beta_3 -} \dots \quad (51)$$

$$\left. \begin{aligned} \alpha_0 b_1 + \beta_0 b_0 &= 0, \\ \alpha_1 b_2 + \beta_1 b_1 + [\alpha_{-1} + \gamma_1] b_0 &= 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} &= 0 \quad (n \geq 2), \end{aligned} \right\} \Rightarrow \beta_0 = \frac{\alpha_0 [\alpha_{-1} + \gamma_1]}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \frac{\alpha_2 \gamma_3}{\beta_3 -} \dots \quad (52)$$

$$\left. \begin{aligned} \alpha_0 b_1 + [\beta_0 + \alpha_{-1}] b_0 &= 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} &= 0 \quad (n \geq 1), \end{aligned} \right\} \Rightarrow \beta_0 + \alpha_{-1} = \frac{\alpha_0 \gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \frac{\alpha_2 \gamma_3}{\beta_3 -} \dots \quad (53)$$

In each one of the the four pairs of truncated solutions, one solution can be expressed in terms of generalized Laguerre polynomials by the Kummer transformation (47) followed by equation (38). Such solutions are: $U_1^0(z)$, $U_2^0(z)$, $U_3^\infty(z)$ and $U_4^\infty(z)$. In the case of finite-series solutions, the other solutions may as well be written as series of generalized Laguerre polynomials, since we have $a = \text{zero}$ or a negative integer in $U(a, b, y)$.

In addition to have changed the notations, we have also reordered the solutions of [1] so that the pairs are obtained by using rules r_2 and r_1 in the same sequence as in section 3. Thereupon we find that the integral relations, the conditions for having terminating series as well as the asymptotic behaviour of each solution are the same as those in the corresponding pairs of section 3.

First pair : $\nu = i\eta$ in $(U_{1\nu}^\infty, U_{1\nu}^0)$

$$\begin{aligned} U_1^\infty(z) &= e^{i\omega z} z^{1+i\eta-(B_2/2)} \sum_{n=0}^\infty b_n^{(1)} (-2i\omega z)^n U(n+1+2i\eta, 2n+2+2i\eta, -2i\omega z), \\ U_1^0(z) &= e^{i\omega z} z^{-i\eta-(B_2/2)} \sum_{n=0}^\infty b_n^{(1)} \left(\frac{B_1}{z}\right)^n U\left(n+i\eta+\frac{B_2}{2}, 2n+2+2i\eta, \frac{B_1}{z}\right). \end{aligned} \quad (54a)$$

$$\begin{aligned} \alpha_n^{(1)} &= \frac{i\omega B_1 [n+1] [n+2+i\eta-(B_2/2)]}{2 [n+1+i\eta] [n+i\eta+(3/2)]}, \\ \beta_n^{(1)} &= B_3 + \left(n+1+i\eta-\frac{B_2}{2}\right) \left(n+i\eta+\frac{B_2}{2}\right) + \frac{\eta\omega B_1 [(B_2/2)-1]}{(n+i\eta)(n+1+i\eta)}, \\ \gamma_n^{(1)} &= \frac{i\omega B_1 [n+2i\eta] [n+(B_2/2)+i\eta-1]}{2 [n+i\eta] [n+i\eta-(1/2)]}. \end{aligned} \quad (54b)$$

$$\text{Integral relation (22): } \Re[(B_2/2) - i\eta - 1] > 0 \text{ and } \Re(B_1/z) > 0. \quad (54c)$$

$$\text{Recurrence relations : } \begin{cases} \text{Eq. (51),} & \text{if } i\eta \neq 0, -1/2; \\ \text{Eq. (52),} & \text{if } i\eta = -1/2; \\ \text{Eq. (53),} & \text{if } i\eta = 0. \end{cases} \quad (54d)$$

$$\text{Finite series, if } (B_2/2) + i\eta = 1 - N \Rightarrow 0 \leq n \leq N - 1. \quad (54e)$$

The asymptotic behavior is given by equations (40a-b).

Second pair : $\nu = i\eta$ in $(U_{2\nu}^\infty, U_{2\nu}^0)$ or $U_{2\nu}^\infty \propto r_2 U_1^\infty$ and $U_{2\nu}^0 \propto r_2 U_1^0$.

$$\begin{aligned} U_2^\infty(z) &= f(z) z^{1+i\eta-(B_2/2)} \sum_{n=0}^\infty b_n^{(2)} (-2i\omega z)^n U(n+1+2i\eta, 2n+2+2i\eta, -2i\omega z), \\ U_2^0(z) &= f(z) z^{-i\eta-(B_2/2)} \sum_{n=0}^\infty b_n^{(2)} \left(-\frac{B_1}{z}\right)^n U\left(n+2+i\eta-\frac{B_2}{2}, 2n+2+2i\eta, -\frac{B_1}{z}\right), \\ f(z) &:= e^{i\omega z + (B_1/z)}. \end{aligned} \quad (55a)$$

$$\begin{aligned} \alpha_n^{(2)} &= -\frac{i\omega B_1 [n+1] [n + (B_2/2) + i\eta]}{2 [n+1+i\eta] [n+i\eta + (3/2)]}, \\ \beta_n^{(2)} &= B_3 + \left(n+1+i\eta - \frac{B_2}{2}\right) \left(n+i\eta + \frac{B_2}{2}\right) + \frac{\eta\omega B_1 [(B_2/2) - 1]}{(n+i\eta)(n+1+i\eta)}, \\ \gamma_n^{(2)} &= -\frac{i\omega B_1 [n+2i\eta] [n - (B_2/2) + i\eta + 1]}{2 [n+i\eta] [n+i\eta - (1/2)]}. \end{aligned} \quad (55b)$$

$$\text{Integral relation (27): } \Re[(B_2/2) + i\eta - 1] < 0 \text{ and } \Re(B_1/z) < 0. \quad (55c)$$

$$\text{Recurrence relations : } \begin{cases} \text{Eq. (51),} & \text{if } i\eta \neq 0, -1/2; \\ \text{Eq. (52),} & \text{if } i\eta = -1/2; \\ \text{Eq. (53),} & \text{if } i\eta = 0. \end{cases} \quad (55d)$$

$$\text{Finite series, if } (B_2/2) - i\eta = N + 1 \Rightarrow 0 \leq n \leq N - 1. \quad (55e)$$

The asymptotic behavior is given by equations (42a-b).

Third pair : $\nu = B_2/2 - 1$ in $(U_{1\nu}^\infty, U_{1\nu}^0)$ or $U_3^\infty \propto r_1 U_1^0$ and $U_3^0 \propto r_1 U_1^\infty$.

$$\begin{aligned} U_3^\infty(z) &= e^{i\omega z} \sum_{n=0}^\infty b_n^{(3)} (-2i\omega z)^n U\left(n + \frac{B_2}{2} + i\eta, 2n + B_2, -2i\omega z\right), \\ U_3^0(z) &= e^{i\omega z} z^{1-B_2} \sum_{n=0}^\infty b_n^{(3)} \left(\frac{B_1}{z}\right)^n U\left(n + B_2 - 1, 2n + B_2, \frac{B_1}{z}\right). \end{aligned} \quad (56a)$$

$$\begin{aligned} \alpha_n^{(3)} &= \frac{i\omega B_1 [n+1] [n + (B_2/2) - i\eta]}{2 [n + (B_2/2)] [n + (B_2/2) + (1/2)]}, \\ \beta_n^{(3)} &= B_3 + n(n + B_2 - 1) + \frac{\eta\omega B_1 [(B_2/2) - 1]}{[n + (B_2/2) - 1] [n + (B_2/2)]}, \\ \gamma_n^{(3)} &= \frac{i\omega B_1 [n + B_2 - 2] [n + (B_2/2) - 1 + i\eta]}{2 [n + (B_2/2) - 1] [n + (B_2/2) - (3/2)]}. \end{aligned} \quad (56b)$$

$$\text{Integral relation (22): } \Re[(B_2/2) - i\eta - 1] > 0 \text{ and } \Re(B_1/z) > 0. \quad (56c)$$

$$\text{Recurrence relations : } \begin{cases} \text{Eq. (51),} & \text{if } B_2 \neq 1, 2; \\ \text{Eq. (52),} & \text{if } B_2 = 1; \\ \text{Eq. (53),} & \text{if } B_2 = 2. \end{cases} \quad (56d)$$

$$\text{Finite series, if } (B_2/2) + i\eta = 1 - N \Rightarrow 0 \leq n \leq N - 1. \quad (56e)$$

The asymptotic behavior is given by equations (44a-b).

Fourth pair : $\nu = 1 - B_2/2$ in $(U_{2\nu}^\infty, U_{2\nu}^0)$ or $U_4^\infty \propto r_2 U_3^\infty$ and $U_4^0 \propto r_2 U_3^0$.

$$\begin{aligned} U_4^\infty(z) &= f(z) z^{2-B_2} \sum_{n=0}^\infty b_n^{(4)} (-2i\omega z)^n U\left(n+2-\frac{B_2}{2}+i\eta, 2n+4-B_2, -2i\omega z\right), \\ U_4^0(z) &= f(z) z^{-1} \sum_{n=0}^\infty b_n^{(4)} \left(-\frac{B_1}{z}\right)^n U\left(n+3-B_2, 2n+4-B_2, -\frac{B_1}{z}\right), \\ f(z) &:= e^{i\omega z + (B_1/z)}. \end{aligned} \quad (57a)$$

$$\begin{aligned} \alpha_n^{(4)} &= \frac{i\omega B_1 [n+1] [n+2-(B_2/2)-i\eta]}{2 [n+2-(B_2/2)] [n+(5/2)-(B_2/2)]}, \\ \beta_n^{(4)} &= -B_3 - (n+1)(n+2-B_2) - \frac{\eta\omega B_1 [(B_2/2)-1]}{[n+1-(B_2/2)] [n+2-(B_2/2)]}, \\ \gamma_n^{(4)} &= \frac{i\omega B_1 [n+2-B_2] [n+1-(B_2/2)+i\eta]}{2 [n+1-(B_2/2)] [n+(1/2)-(B_2/2)]}. \end{aligned} \quad (57b)$$

$$\text{Integral relation (27): } \Re[(B_2/2) + i\eta - 1] < 0 \text{ and } \Re(B_1/z) < 0. \quad (57c)$$

$$\text{Recurrence relations : } \begin{cases} \text{Eq. (51),} & \text{if } B_2 \neq 2, 3; \\ \text{Eq. (52),} & \text{if } B_2 = 3; \\ \text{Eq. (53),} & \text{if } B_2 = 2. \end{cases} \quad (57d)$$

$$\text{Finite series, if } (B_2/2) - i\eta = N+1 \Rightarrow 0 \leq n \leq N-1. \quad (57e)$$

The asymptotic behavior is given by equations (46a-b).

We remark that, in addition to the three possible forms for the recurrence relations, the coefficients of the latter are fractional. Thence, these relations are not well defined when some denominator vanishes. Thus, if $i\eta = \text{negative integer or half-integer} < -1/2$, the coefficients of the first and second pairs are not well defined, but we can form well-defined expressions by using the rule r_3 which changes (η, ω) by $(-\eta, -\omega)$. Similarly, if for some value of B_2 a denominator vanishes for the the third pair, we must consider the solutions of the fourth pair, and vice-versa.

5. DCHE and GSWE: special cases and examples

In this section we examine the two differential equations (10a-b) which share the property of being particular cases of both the DCHE and the GSWE. In such equations, namely,

$$\frac{d^2 W_1}{du^2} + [\theta_0 + \theta_1 \cosh(\kappa u) + \theta_2 \cosh(2\kappa u)] W_1 = 0, \quad (58)$$

$$\frac{d^2 W_2}{du^2} + [\bar{\theta}_0 + \bar{\theta}_1 \sinh(\kappa u) + \bar{\theta}_2 \cosh(2\kappa u)] W_2 = 0, \quad (59)$$

κ is a given constant such that κu is real or pure imaginary and the θ_i ($\bar{\theta}_i$) are constants. Thus there are only three parameters in each equation. If κu is pure imaginary, the first is the Whittaker-Hill equation (WHE) and if κu is real, the modified WHE [14]. In fact, Decarreau, Maroni and Robert [8] have already found that the WHE has that property, whereas we have found that these two equations are special cases of the GSWE [1]. Now we find some normal forms for the DCHE, from one of these we get the particular equations written above and show that they also come from a GSWE with $B_2 = 1$

and $B_1 = -z_0/2$. Finally, we discuss solutions for problems obeying DCHE and GSWE, intending to decide on the best interpretation for each of the special equations (58-59).

5.1. Normal forms for the DCHE

Several normal forms for general Heun's equation and its confluent cases (except the triconfluent equation) were established by Lemieux and Bose [15]. Below, we give some forms suitable for DCHE when it is written as in equation (1).

By performing the substitution

$$U(z) = z^{-B_2/2} e^{B_1/(2z)} F(z) \quad (60a)$$

in equation (1), we find for $f(z)$ an algebraic normal form of the DCHE, namely,

$$\frac{d^2 F}{dz^2} + \left[\omega^2 - \frac{2\eta\omega}{z} + \frac{1}{z^2} \left(B_3 - \frac{B_2^2}{4} + \frac{B_2}{2} \right) + \frac{B_1}{z^3} \left(1 - \frac{B_2}{2} \right) - \frac{B_1^2}{4z^4} \right] F = 0, \quad (60b)$$

where, as before, $B_1 \neq 0$, $\omega \neq 0$. The further transformations

$$z = e^{\lambda u}, \quad F(z) = e^{\lambda u/2} W(u) \Rightarrow W(u) = z^{(B_2-1)/2} e^{-B_1/(2z)} U(z), \quad (61a)$$

where λ is a constant at our disposal, bring the equation to the hyperbolic normal form

$$\frac{d^2 W}{du^2} + \lambda^2 I(u) W = 0, \quad (61b)$$

$$I(u) := - \left[B_1 \left(1 - \frac{B_2}{2} \right) + 2\eta\omega \right] \sinh(\lambda u) + \left[\omega^2 + \frac{B_1^2}{4} \right] \sinh(2\lambda u) + \left[B_1 \left(1 - \frac{B_2}{2} \right) - 2\eta\omega \right] \cosh(\lambda u) + \left[\omega^2 - \frac{B_1^2}{4} \right] \cosh(2\lambda u) + B_3 - \frac{1}{4} (1 - B_2)^2, \quad (61c)$$

to be used soon. The last form we shall need is obtained by the transformations

$$z = \rho^2, \quad U(z) = \rho^{(1-2B_2)/2} e^{B_1/(2\rho^2)} G(\rho) \Rightarrow G(\rho) = z^{(2B_2-1)/4} e^{-B_1/(2z)} U(z) \quad (62a)$$

in equation (1), and this affords another algebraic normal form given by

$$\frac{d^2 G}{d\rho^2} + \left[4\omega^2 \rho^2 - 8\eta\omega + \frac{4}{\rho^2} \left(B_3 - \frac{B_2^2}{4} + \frac{B_2}{2} - \frac{3}{16} \right) + \frac{4B_1}{\rho^4} \left(1 - \frac{B_2}{2} \right) - \frac{B_1^2}{\rho^6} \right] G = 0. \quad (62b)$$

5.2. The common special cases

Equation (61b-c) gives the relations among parameters which lead to equations (58-60) as special cases of the DCHE. For the WHEs we have

$$\begin{aligned} \omega^2 &= -\frac{B_1^2}{4}, \quad 2\eta\omega = -B_1 \left[1 - \frac{B_2}{2} \right] \Rightarrow \\ \frac{d^2 W_1}{du^2} + \lambda^2 \left[B_3 - \frac{1}{4} (1 - B_2)^2 - 4\eta\omega \cosh(\lambda u) + 2\omega^2 \cosh(2\lambda u) \right] W_1 &= 0, \end{aligned} \quad (63)$$

and for the second equation

$$\begin{aligned} \omega^2 &= -\frac{B_1^2}{4}, \quad 2\eta\omega = B_1 \left[1 - \frac{B_2}{2} \right] \Rightarrow \\ \frac{d^2 W_2}{du^2} + \lambda^2 \left[B_3 - \frac{1}{4} (1 - B_2)^2 - 4\eta\omega \sinh(\lambda u) + 2\omega^2 \cosh(2\lambda u) \right] W_2 &= 0. \end{aligned} \quad (64)$$

Now we consider the particular GSWE

$$z(z - z_0) \frac{d^2 U}{dz^2} + \left(-\frac{z_0}{2} + z \right) \frac{dU}{dz} + [B_3 - 2\eta\omega(z - z_0) + \omega^2 z(z - z_0)] U = 0, \quad (65)$$

$[B_1 = -z_0/2, B_2 = 1 \text{ in equation (4)}]$

which has only three constants, since z_0 may be chosen at will, excepting zero. Then, the two special cases follow from the last equation by a change in the independent variable. For the WHEs we have

$$z = z_0 \cosh^2(\sigma u/2), \quad U(z) = W_1(u) \Rightarrow$$

$$\frac{d^2 W_1}{du^2} + \sigma^2 \left[B_3 + \eta\omega z_0 - \frac{1}{8}\omega^2 z_0^2 - \eta\omega z_0 \cosh(\sigma u) + \frac{1}{8}\omega^2 z_0^2 \cosh(2\sigma u) \right] W_1 = 0, \quad (66)$$

and for the second equation

$$z = \frac{z_0}{2} [i \sinh(\sigma u) + 1], \quad U(z) = W_2(u) \Rightarrow$$

$$\frac{d^2 W_2}{du^2} + \sigma^2 \left[B_3 + \eta\omega z_0 - \frac{1}{8}\omega^2 z_0^2 - i\eta\omega z_0 \sinh(\sigma u) - \frac{1}{8}\omega^2 z_0^2 \cosh(2\sigma u) \right] W_2 = 0. \quad (67)$$

Note that the symbols B_i , η and ω are denoting different objects depending on whether they appear in the DCHE or in the GSWE.

In general, solutions for the WHE, when obtained from solutions for the GSWE, are even or odd with respect to the variable u . For example: (i) we have found pairs of even or odd solutions constituted by one solution in series of hyperbolic or trigonometric functions — Arscott's solutions [14, 16]— and another in series of Coulomb wave functions [1], (ii) the Hylleraas and Jaffé type solutions to the GSWE [4] yield only even solutions for the WHE, but the transformation rule T_2 given in equation (6b) of [1] generates new solutions whose limits are odd.

5.3. The examples

We write the one-dimensional time-independent Schrödinger equation for a particle with mass m and energy E as

$$\frac{d^2 \psi(u)}{du^2} + [\mathcal{E} - V(u)] \psi(u) = 0, \quad u := ax, \quad \mathcal{E} := \frac{2mE}{\hbar^2 a^2}, \quad (68)$$

where a is a constant, x the spatial Cartesian coordinate, and ψ must satisfy the regularity conditions

$$\lim_{u \rightarrow \pm\infty} \psi = 0. \quad (69)$$

For the two following examples, we will see that it is convenient to interpret the WHE as a GSWE and the second special equation as a DCHE.

First example: For the symmetric double-Morse potential of Zaslavskii and Ulyanov [17]

$$V(u) = \frac{B^2}{4} \sinh^2 u - B \left(s + \frac{1}{2} \right) \cosh u, \quad B > 0, \quad (70)$$

the Schrödinger equation is a modified WHE. If s is a non-negative integer or a half-integer, the potential (70) is a quasi-exactly solvable (QES) potential in the sense that one part of the energy spectrum stems from finite-series solutions [18]. If this WHE is

considered as a GSWE we find [1]: (i) even and odd quasi-polynomial solutions satisfying the regularity conditions, and (ii) even and odd infinite-series solutions which also satisfy the regularity conditions, provided that we match solutions having the same series coefficients and different radii of convergence. However, if the WHE is interpreted as a DCHE, only finite-series solutions satisfy the regularity conditions, as we will see next.

For the sake of generality, we will get the solutions for the symmetric potential (70) as limits of the solutions for the asymmetric potential [17]

$$V(u) = \frac{B^2}{4} \left(\sinh u - \frac{C}{B} \right)^2 - B \left(s + \frac{1}{2} \right) \cosh u, (B > 0, C \geq 0) \quad (71)$$

for which the Schrödinger equation becomes the DCHE

$$\frac{d^2\psi(u)}{du^2} + \left[\frac{CB}{2} \sinh u + B \left(s + \frac{1}{2} \right) \cosh u - \frac{B^2}{8} \cosh(2u) + \mathcal{E} - \frac{C^2}{4} + \frac{B^2}{8} \right] \psi(u) = 0. \quad (72)$$

Comparing the above equation with equations (61a-c), we find

$$\lambda = 1 \Rightarrow z = e^u, \quad \psi(u) = W(u) = z^{(B_2-1)/2} e^{-B_1/(2z)} U(z), \quad (73a)$$

and the parameters

$$B_1 = \frac{B}{2}, \quad B_2 = 1 + C - 2s, \quad B_3 = \mathcal{E} + \frac{B^2}{8} + s^2 - sC, \quad \omega = i\frac{B}{4}, \quad i\eta = -\frac{C}{2} - \frac{1}{2} - s. \quad (73b)$$

To form regular solutions we identify $U(z)$ with the first and the third pairs of solutions given in section 3. The first pair yields

$$\begin{aligned} \psi_1^\infty(u) &= e^{-\frac{B}{2} \cosh u + (\frac{C}{2} + s)u} \sum_{n=0}^\infty b_n^{(1)} \left(\frac{B}{2} e^u \right)^{-n}, \\ \psi_1^0(u) &= e^{-\frac{B}{2} \cosh u + (\frac{C}{2} + s)u} \sum_{n=0}^\infty b_n^{(1)} U \left(n - 2s, n + 1 - C - 2s, \frac{B}{2} e^{-u} \right), \end{aligned} \quad (74a)$$

with the following coefficients in the recurrence relations (29b) for $b_n^{(1)}$

$$\begin{aligned} \alpha_n^{(1)} &= -(n+1), \\ \beta_n^{(1)} &= -\mathcal{E} - s(s+C) - n(n-C-2s) = \bar{\beta}_n^{(1)} - \mathcal{E}, \\ \gamma_n^{(1)} &= \frac{B^2}{4}(n-2s-1). \end{aligned} \quad (74b)$$

If s is a non-negative integer or half-integer (QES potential), we have $\gamma_{2s+1} = 0$ and thus the series are finite with $0 \leq n \leq 2s$. Thence, the recurrence relations can be written as

$$\begin{pmatrix} \bar{\beta}_0 & \alpha_0 & 0 & \cdots & & 0 \\ & \gamma_1 & \bar{\beta}_1 & \alpha_1 & 0 & \vdots \\ & 0 & \gamma_2 & \bar{\beta}_2 & \alpha_2 & \\ & \vdots & & & & \\ & & & & 0 & \gamma_{2s} & \bar{\beta}_{2s} \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_{2s-1} \\ b_{2s} \end{pmatrix} = \mathcal{E} \begin{pmatrix} b_0 \\ \vdots \\ b_{2s-1} \\ b_{2s} \end{pmatrix}. \quad (75)$$

This system of equations determines $2s+1$ different and real values for \mathcal{E} since, for a eigenvalue problem like this, that is, with a tridiagonal matrix, the following theorem holds (see [14], page 21): ‘if $\alpha_j, \bar{\beta}_j, \gamma_j$ are real and each product $\alpha_j \gamma_{j+1}$ is positive, then the roots corresponding to the equation (75) are all real and different’. In fact, this

remains valid even if both α_j and γ_j are pure imaginaries, since we can take $c_n = i^n b_n$ and put $\alpha'_j = -i\alpha_j$, $\beta'_n = \beta_n$, $\gamma'_j = i\gamma_j$ in the recurrence relations for c_n . The second condition of the theorem stands for the present case because

$$\alpha_j \gamma_{j+1} = -(j+1)(j-2s)B^2/4 > 0,$$

where the inequality follows from the fact that $0 \leq j \leq 2s-1$ for the elements of that matrix. Note that for these quasi-polynomial expansions we can select either the solution $\psi_1^\infty(u)$ or $\psi_3^0(\xi)$, since the hypergeometric functions in the latter reduce to a generalized Laguerre polynomial. The same is true for the solutions that result from the third pair of section 3,

$$\begin{aligned} \psi_3^\infty(u) &= e^{-\frac{B}{2} \cosh u + (\frac{C}{2}-s)u} \sum_{n=0}^{\infty} b_n^{(3)} U\left(n-2s, n+1+C-2s, \frac{B}{2}e^u\right), \\ \psi_3^0(\xi) &= e^{-\frac{B}{2} \cosh u + (\frac{C}{2}-s)u} \sum_{n=0}^{\infty} b_n^{(3)} \left(\frac{B}{2}e^{-u}\right)^{-n}, \end{aligned} \quad (76a)$$

for which the coefficients of the recurrence relations for $b_n^{(3)}$ are

$$\begin{aligned} \alpha_n^{(3)} &= -(n+1), \\ \beta_n^{(3)} &= -\mathcal{E} - s(s-C) - n(n+C-2s) = \bar{\beta}_n^{(3)} - \mathcal{E}, \\ \gamma_n^{(3)} &= \frac{B^2}{4}(n-2s-1). \end{aligned} \quad (76b)$$

The above pairs of solutions are related to one another by means of

$$\left(\psi_1^\infty(C; u), \psi_1^0(C; u)\right) \leftrightarrow \left(\psi_3^0(-C; -u), \psi_3^\infty(-C; -u)\right), \quad (77)$$

that is, these pairs remain invariant under the change $(C, u) \leftrightarrow (-C, -u)$, a property also present in Schrödinger equation (72). Note that we have obtained in [1] one pair of finite-series solutions for equation (72) by using the first pair of solutions given in section 4.2; another pair might be obtained from that by using equation (77) or, alternatively, by using the third pair of solutions given in section 4.2. However, such solutions have recurrence relations with fractional coefficients and, for this reason, they are not well defined for certain integer values of the parameter C , as noted there.

If we suppose that s is not a non-negative integer or half-integer, we can — following Leaver [4], section 8 — match the infinite-series solutions of each pair to get regular eigenfunctions which converge over the entire range of u , since the ψ_1^∞ converge when $\exp u > 0$ and the ψ_1^0 converge when $\exp u < \infty$. In this case, the energy spectrum may be determined from the infinite continued fraction (30). Note also that the conditions to get ψ_i^0 from ψ_i^∞ by an integral transformation are $\Re(C) > 0$ and $\Re(Be^{-u}) > 0$ and, therefore, are assured.

On the other hand, by using the second and fourth pairs of solutions given in section 3 we may form two pairs of infinite-series solutions (even if the potential is QES) but these are not regular when $u \rightarrow -\infty$ due to a multiplicative factor $\exp[-(B/2) \sinh u]$.

For the symmetric case ($C = 0$) the two pairs degenerate to only one

$$\begin{aligned} \psi_1^\infty(u) &= e^{-\frac{B}{2} \cosh u + su} \sum_{n=0}^{\infty} b_n^{(1)} \left(\frac{B}{2}e^u\right)^{-n}, \\ \psi_1^0(u) &= e^{-\frac{B}{2} \cosh u + su} \sum_{n=0}^{\infty} b_n^{(1)} U\left(n-2s, n+1-2s, \frac{B}{2}e^{-u}\right) \\ &= (B/2)^{2s} e^{-\frac{B}{2} \cosh u - su} \sum_{n=0}^{\infty} b_n^{(1)} \left(\frac{B}{2}e^{-u}\right)^{-n} \quad [\text{see equation (47)}] \end{aligned} \quad (78a)$$

and, in the recurrence relations (75), we have

$$\alpha_n^{(1)} = -(n+1), \quad \beta_n^{(1)} = -\mathcal{E} - s^2 - n(n-2s) = \bar{\beta}_n^{(1)} - \mathcal{E}, \quad \gamma_n^{(1)} = \frac{B^2}{4}(n-2s-1) \quad (78b)$$

For finite-series (QES potential) the two solutions are convergent and regular for $u \in (-\infty, \infty)$, but are neither even nor odd with respect to u . However, we can form even and odd eigenfunctions by taking the linear combinations

$$A_1\psi_1^0(u) + A_2\psi_1^\infty(u) = e^{-(B/2)\cosh u} \sum_{n=0}^{2s} b_n^{(1)} \left[A_1 \left(\frac{B}{2} \right)^{2s} e^{(n-s)u} + A_2 e^{-(n-s)u} \right],$$

and choosing the constants A_1 and A_2 such that we have series of $\cosh[(n-s)u]$ and $\sinh[(n-s)u]$, respectively. Note the absence of infinite-series solutions for this QES potential, contrary to the solutions resulting from the GSWE [1].

Second example. This example also deals with a QES potential, now giving a differential equation of the second type. In contrast with the first example, we find that: (i) if the equation is treated as a DCHE, only infinite-series solutions satisfy the regularity conditions, (ii) if the equation is treated as a GSWE, there is no regular solutions.

The potential is

$$V(u) = \frac{B^2}{4} \sinh^2 u - \left(s + \frac{1}{2} \right) B \sinh u, \quad B > 0, \quad s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \quad (79)$$

and thus the Schrödinger equation (68) reads

$$\frac{d^2\psi(u)}{du^2} + \left[\mathcal{E} + \frac{B^2}{8} + \left(s + \frac{1}{2} \right) B \sinh u - \frac{B^2}{8} \cosh(2u) \right] \psi(u) = 0. \quad (80)$$

Comparing this equation with equations (61a) and (64), we get

$$\lambda = 1 \Rightarrow z = e^u, \quad \psi(u) = W(u) = z^{(B_2-1)/2} e^{-B_1/(2z)} U(z), \quad (81a)$$

together with the following expressions for the parameters

$$B_1 = -\frac{B}{2}, \quad B_2 = 1 - 2s, \quad B_3 = \mathcal{E} + \frac{B^2}{8} + s^2, \quad i\omega = -\frac{B}{4}, \quad i\eta = -\frac{1}{2} - s. \quad (81b)$$

Now, using the second and fourth pairs of solutions given section 3, we obtain two pair of infinite-series solutions for $\psi(u)$. The first is

$$\begin{aligned} \psi_2^\infty(u) &= e^{-\frac{B}{2} \cosh u + su} \sum_{n=0}^{\infty} b_n^{(2)} \left(\frac{B}{2} e^u \right)^{-n}, \\ \psi_2^0(u) &= e^{-\frac{B}{2} \cosh u + su} \sum_{n=0}^{\infty} b_n^{(2)} U \left(n+1, n+1-2s, \frac{B}{2} e^{-u} \right), \end{aligned} \quad (82a)$$

with

$$\alpha_n^{(1)} = n+1, \quad \beta_n^{(1)} = \mathcal{E} + s^2 + n(n-2s), \quad \gamma_n^{(1)} = -\frac{B^2}{4}n. \quad (82b)$$

in the recurrence relations (29b). The second pair is given by

$$\begin{aligned} \psi_4^\infty(u) &= e^{-\frac{B}{2} \cosh u + (1+s)u} \sum_{n=0}^{\infty} b_n^{(4)} U \left(n+1, n+3+2s, \frac{B}{2} e^u \right), \\ \psi_4^0(\xi) &= e^{-\frac{B}{2} \cosh u + (1+s)u} \sum_{n=0}^{\infty} b_n^{(4)} \left(\frac{B}{2} e^{-u} \right)^{-n}, \end{aligned} \quad (83a)$$

and has

$$\alpha_n^{(4)} = n + 1, \quad \beta_n^{(4)} = \mathcal{E} + (s + 1)^2 + n(n + 2 + 2s), \quad \gamma_n^{(4)} = -\frac{B^2}{4}n. \quad (83b)$$

in the recurrence relations (29b). In these two pairs of nonterminating series we have to match the solutions in order to assure regularity over the entire interval for u . Now, the conditions to derive ψ_i^0 from ψ_i^∞ by an integral transformation are $\Re(2s + 1) > 0$ and $\Re(Be^{-u}) > 0$. Quasi-polynomial solutions result from the first and third pairs of section 3 but they are not regular due to the presence of the factor $\exp[-(B/2)\sinh u]$, as in the previous example.

We have not found any regular solution by describing this problem by a GSWE. In effect, comparing equations (80) and (67), we find

$$\sigma = z_0 = 1, \quad z = \frac{i}{2}\sinh u + \frac{1}{2}, \quad \omega = B, \quad i\eta = -s - \frac{1}{2}, \quad B_3 = \mathcal{E} + \frac{B^2}{4} - i\left(s + \frac{1}{2}\right)B.$$

However the solutions for the GSWE have the factor $\exp(\pm i\omega z) = \pm B(-\sinh u + i)/2$ which diverges when $u \rightarrow \infty$ or $u \rightarrow -\infty$. As an illustration, we use the pair of solutions given in equation (42) of [1] and obtain

$$\begin{aligned} \psi_1 &= e^{-(B\sinh u)/2} \sum_{n=0}^{\infty} b_n^{(1)} F\left(-n, n; \frac{1}{2}; \frac{1 - i\sinh u}{2}\right), \\ \tilde{\psi}_1 &= e^{-(B\sinh u)/2} \sum_{n=0}^{\infty} b_n^{(1)} (B\sinh u - iB)^n U(n - s, 2n + 1; B\sinh u - iB), \end{aligned}$$

where the recurrence relations have the form given in equations (52), with

$$\alpha_n^{(1)} = \frac{iB}{2}(n + s + 1), \quad \beta_n^{(1)} = -n^2 - \mathcal{E} - \frac{B^2}{4}, \quad \gamma_n^{(1)} = -\frac{iB}{2}(n - s - 1).$$

Hence it follows that, if s is a non-negative integer, we have finite series and, if s is a half-integer, we have infinite series, but none satisfies the regularity conditions.

We note that the potential (79) was obtained from the potential (70) by means of the change $[s + (1/2)]B \cosh u \rightarrow [s + (1/2)]B \sinh u$, but it can also be derived from the potential

$$V(u) = V_1 \sinh^2 u + V_2 \sinh u + \frac{V_3 \sinh u + V_4}{\cosh^2 u}, \quad u \in (-\infty, \infty) \quad (84)$$

for which the Schrödinger equation is a GSWE [15]. A QES version of (84) is given in [19] and from that we can obtain (79) by taking $V_3 = V_4 = 0$.

To finalize we mention two other problems which are reducible to a DCHE in one or another of the algebraic normal forms. They are related with the three-dimensional radial Schrödinger equation

$$\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \left[\mathcal{E} - \frac{l(l+1)}{r^2} - V(r) \right] R = 0, \quad \mathcal{E} := \frac{2mE}{\hbar^2}, \quad (85)$$

for a particle with mass m and energy E . A normal form for this equation is

$$\frac{d^2 H(r)}{dr^2} + \left[\mathcal{E} - \frac{l(l+1)}{r^2} - V(r) \right] H(r) = 0, \quad H(r) := rR(r). \quad (86)$$

Then, for the inverse fourth-power potential

$$V(r) = V_1 r^{-1} + V_2 r^{-2} + V_3 r^{-3} + V_4 r^{-4}, \quad (87)$$

equation (86) assumes the normal form (60b) for the DCHE with $z = r$, whereas for the even-power potential

$$V(r) = V_1 r^2 + V_2 r^{-2} + V_3 r^{-4} + V_4 r^{-6} \quad (88)$$

it assumes the normal form (62b) with $\rho = r$. Solutions have been proposed for problems like these [20-23], but it would be interesting to study such problems from the viewpoint of the DCHE since the transformations rules and integral relations allow us to obtain new solutions from a known one. We could also check whether the solutions presented here are useful for that purpose.

6. Concluding remarks

We have found integral relations for solutions of the double-confluent Heun equation and combined them with transformation rules in order to obtain the group of solutions given in section 3. In section 4, the integral relations (22) and (27) have also been used to connect expansions in series of Coulomb wave functions. The solutions have been displayed in pairs $(U_i^\infty(z), U_i^0(z))$, where $U_i^\infty(z)$ and $U_i^0(z)$ converge for $|z| > 0$ and $|z| < \infty$, respectively. The solutions in each pair have the same series coefficients just because $U_i^0(z)$ comes from $U_i^\infty(z)$ by an integral relation, whose validity conditions (18a-b) were satisfied due to the choice of an integration contour that excluded the point $t = 0$, where the solution $U_i^\infty(t)$ does not converge.

Comparing each pair of solutions given in section 3 with the corresponding pair given in section 4.2, we find (i) the same integral relations between the solutions, (ii) the same asymptotic behaviour for the solutions, (iii) the same conditions for quasi-polynomial solutions, and (iv) the possibility of writing one solution as a generalized Laguerre polynomial. The last property may be important if we need to normalize solutions. Note, however, that in section 3 there is only one form for the recurrence relations, whereas in section 4.2 there are three possible forms with fractional coefficients which are not well defined when a denominator vanishes.

In section 5 we have given normal forms for the DCHE and analysed equations (10a-b) that have the common property of being particular cases of both the DCHE and the GSWE. We have as well looked for solutions to the Schrödinger equation for two quasi-exactly solvable hyperbolic potentials. For the symmetric Zaslavskii-Ulyanov potential we have a modified WHE, for which we have established regular quasi-polynomial and infinite-series solutions by considering this WHE as a GSWE [1], but only quasi-polynomial solutions by treating it as a DCHE. For the potential (79), in the second example, the Schrödinger equation is an equation of the second special type, for which we have found regular infinite-series solutions by interpreting the equation as a DCHE; however we have not found any regular solution by considering it as a GSWE. For infinite-series solutions, which have been formed by joining solutions belonging to a same pair, the energy spectra may be computed from infinite continued fractions, for instance.

The results of the preceding paragraph suggest that we must interpret the WHE (10a) as a GSWE, and equation (10b) as a DCHE. However, it is necessary to consider other problems such as the time dependence of the Dirac equation in radiation-dominated Friedmann-Robertson-Walker spacetimes (see section 2.2.1 of [1]). The differential equations for these problems have no free parameters, which implies that we have to deal

with double-sided series possessing a phase parameter, as those of section 4.1 for equation (10b) or the solutions of section 2.2 of [1] for the WHE.

In section 5 we have also discussed solutions to the Schrödinger equation with an asymmetric Zaslavskii-Ulyanov potential, but now using the new solutions found in section 3. The regular solutions are quasi-polynomial and do not exclude integral values for the parameter C , contrary to the solutions constructed on the basis of the expansions given in section 4.2 [1]. Despite this advantage, the obtainment of regular infinite-series solutions for this problem remains unsolved.

Finally, we have called attention to some singular radial potentials for which the Schrödinger equation leads to DCHEs, but we have not tried to solve these equations. Another related problem concerns the solutions of the Schrödinger equation for the QES potentials derived recently by Bagchi and Ganguly [24], since for their hyperbolic potentials we will find modified WHEs, and for trigonometric potentials, WHEs. Considering these equations as particular cases of the GSWE, we may expect to find quasi-polynomial and infinite-series solutions as in the first example of section 5.

In the Appendix we have rewritten some integrals in a form suitable for use in section 3 and 4. Thus, excepting a correction to a misprint in a table of integrals, this appendix does not contain anything original.

Appendix. Integrals used in sections 3 and 4

The first equation is an integral representation for the irregular confluent hypergeometric function $U(a, b, z)$ given by [12]

$$\int_1^\infty e^{-yt}(t-1)^{\alpha-1}t^{\beta-\alpha-1}dt = \Gamma(\alpha)e^{-y}U(\alpha, \beta, y), \quad [\Re\alpha > 0, \Re y > 0]. \quad (\text{A1})$$

The two following integrals are usually given in terms of irregular Whittaker functions $W_{\kappa, \mu}(z)$. By convenience, we have reexpressed them in terms of irregular confluent hypergeometric functions by using the relation [13]

$$W_{\kappa, \mu}(y) = e^{-y/2}y^{\mu+(1/2)}U\left(\frac{1}{2} - \kappa + \mu, 2\mu + 1, y\right).$$

The first of these integrals is

$$\begin{aligned} & \int_1^\infty e^{-ay}(y-1)^{\mu-1}U\left(\frac{1}{2} - \kappa - \lambda, 1 - 2\lambda, ay\right)dy \\ &= \Gamma(\mu)e^{-a}a^{-\mu}U\left(\frac{1}{2} - \kappa - \lambda, 1 - 2\lambda - \mu, a\right), \quad [\Re\mu > 0, \Re a > 0], \end{aligned} \quad (\text{A2})$$

which results from [25]

$$\int_1^\infty e^{-ay/2}(y-1)^{\mu-1}y^{\lambda-\frac{1}{2}}W_{\kappa, \lambda}(ay)dy = \Gamma(\mu)e^{-a/2}a^{-\mu/2}W_{\kappa-\frac{\mu}{2}, \lambda+\frac{\mu}{2}}(a), \quad [\Re\mu > 0, \Re a > 0].$$

Note a misprint on page 867 of [26] where we have $W_{\kappa-(\mu/2), \lambda-(\mu/2)}(a)$ on the right-hand side of the above integral. The other integral is

$$\begin{aligned} & \int_1^\infty e^{-ay}(y-1)^{\mu-1}y^{\kappa+\lambda-\mu-\frac{1}{2}}U\left(\frac{1}{2} + \lambda - \kappa, 2\lambda + 1, ay\right)dy \\ &= \Gamma(\mu)e^{-a}U\left(\frac{1}{2} + \mu - \kappa + \lambda, 1 + 2\lambda, a\right), \quad [\Re\mu > 0, \Re a > 0], \end{aligned} \quad (\text{A3})$$

which is equivalent to [26]

$$\int_1^\infty e^{-ay/2}(y-1)^{\mu-1}y^{\kappa-\mu-1}W_{\kappa, \lambda}(ay)dy = \Gamma(\mu)e^{-a/2}W_{\kappa-\mu, \lambda}(a), \quad [\Re\mu > 0, \Re a > 0].$$

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